AD-774 161

NUMERICAL SIMULATION OF TURBULENCE

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Stanford Research Institute

Prepared for:

Defense Advanced Research Projects Agency

November 1973

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER
AD 774161

4. TITLE (and Subtitle)
NUMERICAL SIMULATION OF TURBULENCE

5. TYPE OF REPORT & PERIOD COVERED
Technical Report JSR-73-3

6. PERFORMING ORG. REPORT NUMBER
SRI Project 3000

7. AUTHOR(s)
K. M. Case F. J. Dyson E. A. Frieman
C. E. Grosch F. W. Perkins

8. CONTRACT OR GRANT NUMBER(s)
Contract DAHC15-73-C-0370

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Stanford Research Institute
Menlo Park, California

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Defense Advanced Research Projects Agency
1400 Wilson Boulevard
Arlington, Virginia 22209

12. REPORT DATE
November 1973

13. NO. OF PAGES
63

14. MONITORING AGENCY NAME & ADDRESS (if diff. from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Turbulence solutions
Numerical simulation of turbulence
Transititon flow calculations
Turbulence calculations by use of large digital computers
Computational methods for solutions for turbulence

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The question considered is whether now is an appropriate time for a direct numerical simulation of turbulence, with particular emphasis on the use of the Illiac IV.

First, it should be emphasized that the use of numerical simulation as a general tool for turbulence calculations is out of the question. To resolve all scales of motion when the Reynolds number is R requires following on the order of $R^{9/4}$ independent variables in time, and on the order of $R^3 \ell nR$ arithmetic operations

DD FORM 1 JAN 73 1473
EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Date Entered)

19. KEY WORDS (Continued)

20 ABSTRACT (Continued)

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The realistic goal of practical turbulence theory is to achieve approximate equations for mean quantities. These equations, even though complicated looking, should be such that the quantities involved are all smoothly varying functions.

Numerical simulation can contribute to this goal in two ways:

(1) It can provide "benchmark" problems against which approximations can be tested. (The attack on these problems can be regarded as a precision experiment.) This report discusses particular problems of this kind.

(2) It can help to provide theoretical insight into the nature of turbulence. (For example, numerical experiments are possible that are not feasible in the laboratory.) Hopefully, universal features will be found and then approximate methods can be developed.

For three-dimensional problems it is shown that the Reynolds number of problems that can be treated, even by the new generation of computers, is rather limited ($R \lesssim 10^4$ to 4×10^4). The Illiac IV is especially restricted in treating turbulence problems. It has a relatively small fast memory. For the problems of interest, there are many variables and the interactions are nonlocal--i.e., quantities to be computed at a point depend on ones at all other points. This does not mean that important problems cannot be simulated. Transition from laminar to turbulent flow and the behavior at low Reynolds number above transition are of interest. Some of these can be treated with existing computers (the IBM 360/195 and the CDC 7600). However, it appears unlikely that a major breakthrough will necessarily result from the calculations. But even these calculations may serve to disprove some approximation methods.

In the somewhat less realistic case of two dimensions, the situation is slightly different. Calculations already done show a tendency toward Reynolds-number independence of macroscopic features. If confirmed by the results of the higher Reynolds-number calculations (made possible with the new computers) this could be very significant. (The caveat about the Illiac IV still applies.)

Finally, we consider what computer capability is needed to materially change the above picture, and sketch how this could possibly be achieved and at what cost.

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Technical Report JSR-73-3

November 1973

NUMERICAL SIMULATION OF TURBULENCE

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PROJECTS AGENCY
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ARLINGTON, VIRGINIA 22209

CONTRACT DAHC15-73-C-0370

SRI Project 3000

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CONTENTS

TABLES

I INTRODUCTION

Turbulent flows play a major role in many problems of great practical interest. Among these are flow in pipes (industrial applications), wakes (reentry vehicles, submarines), boundary layers (separation, transition), meteorology, airplane designs, and nuclear explosions. Unfortunately, even at this date our understanding of turbulence is severely limited. Qualitative features are known as are some general predictions (based usually on some reasonable assumptions and some dimensional analysis). Practical problems are usually treated by phenomenological equations. The range of validity of these is either unknown or known to be restricted to situations close to those in which the phenomenological constants are determined.

Here the question is considered as to whether the advent of a new generation of computers (Illiac IV, Star, ASC) makes feasible a numerical attack on turbulence by direct integration of the Navier-Stokes equations.

It is concluded that, while some useful problems can (and should) be done, it does not seem that a fundamental breakthrough is perceivable. This statement is particularly true for the Illiac IV because of memory limitations.

This should be understood in the following context:

(1) Presumably most practical problems are so complex, both geometrically and physically, that we do not expect a direct solution to be possible from first principles.

(2) The main obstacle to numerical solution of the Navier-Stokes equation lies in the fact that when there is turbulence we have interesting and important motions on many different scales (both space and time). In particular, the larger

the Reynolds number the smaller are the scales that need be considered. Clearly, for any computer there will be some Reynolds number beyond which the computation of solutions is not reasonable. We do not expect to find asymptotic behaviors numerically, but rather hope that the numerics will be a guide to an asymptotic theory.

Accordingly, we phrase the question as follows: Are there problems of interest that can now be treated that were previously inaccessible to direct numerical solution? By "interest" we mean either:

(1) They are "benchmark" problems: The results should be verifiable by experiments. Assuming agreement is achieved, one then would have a much more detailed description than can ever be found from experiment. (For example, one could calculate as many correlation functions and their transforms as desired.) These would then be tests to which approximation methods would be subject.

(2) The results give theoretical insight: For example, is the transition from laminar to turbulent flow as Reynolds number increases due to successively more instabilities arising? Is our picture of energy and vorticity cascade correct? Alternatively, a better understanding of these effects can be obtained by varying boundary conditions in either wave number or physical space. Here the calculations are "experiments" which are either impossible or very expensive to do in a laboratory.

Essentially what we would like to do is:

(1) Determine whether there are indeed universal features of turbulence.

(2) If there are such, can we develop equations which describe the average flows without requiring the resolution of the fine scales of the turbulent motion?

To get some idea of the barriers encountered, we note that for a problem in d dimensions with N degrees of freedom in each linear dimension, the number of arithmetic operations to be done is on the order of

$$aN^{d+1}\ln_2 N + bN^{d+1}$$

where a and b $\sim 10^3$. As a rule of thumb, we need $N \sim R^{3/4}/10$ where R is the Reynolds number. Two things stand out: For a given computer, much more can be done in two dimensions than three. To significantly increase the Reynolds numbers for which one can calculate (say by a factor of two to ten) requires a huge increase in computer capability. (This is particularly true in three dimensions.)

However, it does seem that some interesting problems might now be done. The estimates made below are based primarily on the characteristics of the Illiac IV and are admittedly somewhat optimistic. While each computer has its own limitations and advantages it is not thought that the situation would be radically different for other computers of the new generation. (As mentioned above, the Illiac IV does have memory problems that may prove intractable.)

Problems that might be approached are:

(1) The behavior of flows as the Reynolds number is gradually increased above the critical value in Poisseuille flow in channels and pipes, and Couette flow in channels and between rotating cylinders.

(2) Turbulent flows in pipes and channels for $R \sim 10^4$.

(There are many experimental results available for these two classes or problems.)

(3) Problems of homogeneous isotropic turbulence. These would be primarily numerical experiments--to enlarge theoretical understanding.

(4) Much higher Reynolds numbers in two dimensions can be investigated. This is of interest theoretically and has practical use for atmospheric problems.

(5) Problems involving turbulent flows in boundary layers--say over a flat plate. This is something that probably should be one of the last to be considered. The difficulty is that to treat these numerically one needs to put in some (perhaps very reasonable) assumptions as to the behavior of the flows at large distance from the flow. The results might merely reflect these assumptions and not the correct physics.

Some important questions will not be resolved by such calculations. The existence of an inertial range and Kolmogoroff $k^{-5/3}$ law (or proposed modifications) would not be seen in three dimensions. (However, the analogous k^{-3} might be observed for two-dimensional problems.) The interesting questions of intermittency would also not be particularly clarified.

What can one hope to be the conclusions from such calculations? First, we might verify the expectation that on small scales turbulence is essentially universal--i.e., rather independent of the large scale structure. Second, one might check previous calculations which suggest that at sufficiently high Reynolds numbers the macroscopic features of a flow are rather Reynolds-number independent. From numerical experiments, new methods for treating sub-grid scale features of flow may be suggested. These (and existing) methods can then be tested by the calculations.

It may be noted that compressible problems have not been mentioned. This is, of course, not because they are uninteresting. Indeed, some methods of calculation exploit compressibility. However, there is even less understanding of turbulence in the compressible case. In light of our expectation that a numerical program is, at least in part, to extend theoretical understanding, it seems best to start where we feel we now have the most knowledge.

If a numerical program is undertaken (say on Illiac IV) it should go forward to begin with at a rather moderate pace. First an attempt should be made to see how a program might be written for some simple problem. It seems clear from our estimates that problems of interest will strain the capability of the Illiac as it is expected to be in the next year or so. While we have tried to anticipate problems that might arise due to pecularities of the Illiac, there may be additional

ones which rear their heads when a detailed program is attempted. In particular it seems very likely that the computation scheme that should be chosen may well be determined by properties of the machine. For example, input-output times and memory capacity may well dictate a scheme that requires somewhat more arithmetic operations than a perhaps otherwise optimum method.

After a program has been written, numerous tests would need to be performed. These might be other (smaller) calculations done in two dimensions and also on homogeneous isotropic turbulence in three dimensions. The standard color-cone problem and the Taylor-Green problem may also be used.

It is also thought that the whole program should not be allocated to a single contractor, and there should be an outside advisory group. The reasoning here is that since the hope is to gain insight as the program progresses, one would expect the selection of succeeding problems to be determined by what has been already found.

In the following sections we amplify and sketch the reasoning for the above remarks. Section II briefly describes the present status of turbulence theory and experiment. Section III describes calculations which have been done and those that might be done. Some general features of computation procedures for numerical simulation of turbulence are described in Section IV. Specific details of the Illiac IV, which determine what calculations might be done and how the choice of procedures may be restricted, are discussed in Section V. Some estimates of computing time are given there. Problems of interest that could be done, and how they might form a program of work on the Illiac, are given in Section VI.

A brief consideration on the design of a computer that might be specific for turbulence problems is given in Appendix A.

II TURBULENCE

A. Status of Theory

Most flows in nature are turbulent--i.e., even if they start with a relatively smooth laminar profile they rapidly develop a random and fluctuating appearance. Qualitative understanding is readily obtained from the Navier-Stokes equations for incompressible flow:

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla\right) \underset{\sim}{v} = -\nabla P + \nu \nabla^2 \underset{\sim}{v} \tag{1}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{2}$$

where $\underset{\sim}{v}(\underset{\sim}{r},t)$ is the velocity field, $P(\underset{\sim}{r},t)$ is the pressure and ν is the kinematic velocity. Typical values at STP are (in square centimeters per second) $\nu_{air} = 0.145$, $\nu_{H_2O} = 0.011$, $\nu_{Hg} = 0.00116$. Let us estimate the ratios of the non-linear terms to the viscous terms--i.e.,

$$R \sim \frac{\underset{\sim}{v} \cdot \nabla \underset{\sim}{v}}{\nu \nabla^2 v} \quad .$$

If V is a typical value of the velocity and L a measure of the scale over which the variations are occurring, we see

$$R = \frac{VL}{\nu} \quad .$$

In most cases this is a large number. For example, for water flowing in a pipe of radius 10 cm with velocity 10 cm/s, $R \simeq 10^4$. For air

flowing at 10 m/hr over a structure of radius 100 meters, $R \simeq 4 \times 10^7$. Under these circumstances one might assume that the viscous terms are completely negligible. However, let us note that the convective term $(\underset{\sim}{v} \cdot \nabla \underset{\sim}{v})$ is nonlinear. Expressed in wave number space, this shows that the various Fourier components are coupled together. Even if we start with only small wave numbers the coupling will bring in higher wave numbers. For these wave numbers the viscous term is larger. Eventually it will even dominate the convective term, and convert the flow energy to heat. Thus we have the following picture for flows at large Reynolds number: Energy inserted at small wave numbers cascades to higher wave numbers due to the nonlinear terms. Eventually viscosity dominates and the energy is dissipated. The random fluctuating behavior seen at large Reynolds numbers is then due to the fact that high wave numbers (i.e., high Fourier components) are excited.

An interesting description of the onset of turbulence has been given by Landau:[1*] Below a certain critical Reynolds number the laminar flow is stable. Slightly above the critical Reynolds, any small perturbation will grow slowly until the solution is the original laminar flow plus a small, sinusoidally varying contribution with phase determined by the perturbation. For larger R the separation into laminar and sinusoidal becomes no longer meaningful. We then have a periodic flow, but not merely simply periodic. As R increases further we get a new instability and eventually a doubly periodic flow with two arbitrary phases (essentially determined by the usually undescribable perturbation). Proceeding to higher and higher R we get successively more and more multiple periodic motions with more and more arbitrary phases. Hence the randomness of the motion.

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* References are listed at the end of this report.

Now for practical purposes we are not interested in all the details of the fluctuating motion, but rather in certain average values. (For the experimentalist these tend to be either spatial or temporal averages. The theorist, whose point of view we take, thinks in terms of ensemble averages, and implicitly an ergodic theorem.) With bars denoting the appropriate averages it is conventional to divide the flow into its average and fluctuating parts--i.e., $\underset{\sim}{v} = \bar{\underset{\sim}{v}} + \underset{\sim}{v}'$. Taking averages of Eqs. (1) and (2) gives

$$\frac{\partial}{\partial t}\,\bar{v}_i + \bar{v}_j\,\frac{\partial}{\partial x_j}\,\bar{v}_i$$

$$= -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{v}_i}{\partial x_j} - \overline{v_i' v_j'}\right) \qquad \nabla \cdot \bar{\underset{\sim}{v}} = 0 \quad . \tag{3}$$

These are perfectly nice equations for $\bar{\underset{\sim}{v}}$, but involve, however, the Reynolds stress, $\overline{v_i' v_j'}$. Thus, we have more unknowns than equations. Of course, we can obtain equations for the Reynolds stress by multiplying the original Eqs. (1) and (2) by $\underset{\sim}{v}$ and then averaging. But then quantities like $\overline{p' v_i'}$ and $\overline{v_i' v_j' v_k'}$ occur. Again more unknowns than equations. Proceeding to higher order we find the same situation at each step. Breaking this chain is the closure problem. Almost all approximation methods are based on some assumption as to how this can be done. For example, the way the Reynolds stress occurs in Eq. (3) suggests replacing that term by one like that due to the kinematic viscosity but with a fictitious effective viscosity. This eddy viscosity could, of course, depend on $\underset{\sim}{r}$ and $\bar{\underset{\sim}{v}}$, and even more complicated quantities to be discussed later. Alternatively, assumptions have been made about higher order correlation coefficients. It may be noted that the resulting equations are then frequently more complicated looking than

Eqs. (1) and (2). However, they are hopefully for smoothly varying functions and therefore easier to use in calculations.

What can we say about the Reynolds stress tensor using our qualitative ideas and some dimensional analysis? If the ideas about cascading in wave number space are right we expect that

$$\overline{v_i'(\underset{\sim}{r},t)v_j'(\underset{\sim}{r}+\underset{\sim}{s},t)}$$

is rather independent of $\underset{\sim}{r}$ for small $\underset{\sim}{s}$. Let $\varphi_{ij}(\underset{\sim}{k})$ be the Fourier transform with respect to $\underset{\sim}{s}$--i.e.,

$$\varphi_{ij}(\underset{\sim}{k}) = (2\pi)^{-3} \iiint_{-\infty}^{\infty} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{s}} \overline{v_i'(\underset{\sim}{r},t)\, v_j'(\underset{\sim}{r}+\underset{\sim}{s},t)}\, d^3\underset{\sim}{s} \quad .$$

The trace of this quantity can be interpreted as the kinetic energy at a given wave number. Indeed,

$$\overline{v_i'(\underset{\sim}{r},t)\, v_j'(\underset{\sim}{r},t)} = 3u^2 = \iiint \varphi_{ij}(\underset{\sim}{k})d^3k \quad .$$

Defining

$$E(k) = \tfrac{1}{2} \iint \varphi_{ii}(k)\, k^2\, d\Omega_{\underset{\sim}{k}}$$

(the integral is over a spherical shell of radius k), we have

$$\int_0^\infty E(k)\, dk = 3/2\, u^2 = \text{kinetic energy per unit mass.}$$

Consider the following simple model: We have a stationary state with turbulent energy, u^2, being introduced over a macroscopic length, ℓ. By the cascade process, this is then transferred to higher wave numbers. A measure of the time for this to take place is ℓ/u. Hence

the rate at which energy is fed into the cascade is $u^2/(\ell/u) = u^3/\ell$. At sufficiently high k this energy is dissipated by viscosity at a rate ϵ which (if we are to have stationarity) must be $\epsilon = u^3/\ell$. In the dissipation region the only relevant parameters are presumably the kinematic viscosity ν, and the dissipation rate ϵ. The appropriate time and space scales relevant are then those which can be formed from these quantities. They are the Kolmogoroff[2] microscales of time $\tau = (\nu/\epsilon)^{\frac{1}{2}}$, distance $\kappa = (\nu^3/\epsilon)^{\frac{1}{4}}$. From these one can form the velocity scale $v = (\nu\epsilon)^{\frac{1}{4}}$. Now let us look at ranges of k which are considerably larger than $1/\ell$. One expects E(k) to be a function only of ϵ, ℓ, and κ. Then on dimensional grounds

$$E(k) = u^2 \ell F(k\ell, k\kappa) \tag{4a}$$

or alternatively

$$E(k) = v^2 \kappa C(k\ell, k\kappa) \quad . \tag{4b}$$

Note: The various time, distance and velocity scales are all related to the Reynolds number. Thus

$$\frac{v}{u} \sim R^{-\frac{1}{4}} \quad , \quad \frac{\kappa}{\ell} \sim R^{-\frac{3}{4}} \quad , \quad \frac{\tau}{\ell/u} \sim R^{-\frac{1}{2}} \quad . \tag{5}$$

Now let us suppose that we have a region of k such that:

$$k\ell \gg 1, \text{ and simultaneously } k\kappa \ll 1 \quad . \tag{6}$$

From Eq. (4a) we then expect

$$E(k) \sim u^2 \ell F(k\ell, 0)$$

and from Eq. (4b)

$$E(k) \approx v^2 \kappa \, G(\infty, k\kappa)$$

Equating this then shows $v^2 \kappa \, G(\infty, k\kappa)$ is independent of ν. This can only be true if $G(\infty, k\kappa) = \alpha \, (k\kappa)^{-5/3}$ with α some universal constant.

Therefore we expect that there may be a region of k such that[2]

$$E(k) = \alpha \, \varepsilon^{\frac{2}{3}} \, k^{-5/3} \quad . \tag{7}$$

Remarkably enough there are geophysical flows in which a region of E(k) of this form is not inconsistent with the data (with $\alpha \sim 1.5$). We note though, that this requires there be a region of k far from both those k directly affected by macroscopic variations and much less than the k region subject to significant dissipation. Further, the inequalities of Eq. (6) must hold. Combining these we note a necessary condition for such a region to exist is

$$k\kappa \ll 1 \ll k\ell \quad \text{or} \quad \frac{\ell}{\kappa} \sim R^{\frac{3}{4}} \ggg 1 \quad .$$

That is, the Reynolds number must be very large.

Similar dimensional arguments[3] are extremely useful in obtaining estimates of the velocity profiles in the vicinity of walls (we will need a knowledge of this behavior in the following. Consider the idealized case of a steady plane-parallel flow of fluid moving in the x direction in the space between moving rigid walls at $z = 0$ and $z = h$ in the absence of a mean pressure gradient. The x-component of Eq. (2) gives

$$\nu \frac{d^2\bar{v}_x}{dz^2} - \frac{d}{dz}\overline{v'_x v'_z} = 0 \quad ;$$

i.e., $$\tau(z) = \nu \frac{d\bar{v}_x}{dz} - \overline{v'_x v'_z} = \text{constant} = \tau_o \quad . \tag{8}$$

The average characteristics of the flow can depend only on the shear stress τ_o, kinematic viscosity, and the distances z, h-z. It is conventional to introduce

$$u^* = \sqrt{\tau_o} \quad .$$

Then dimensionally we expect

$$\bar{v}_x(z) = u^* f\left(\frac{zu^*}{\nu}, \frac{(h-z)u^*}{\nu}\right) \quad .$$

Now, very much as in the derivation of the Kolmogoroff spectrum, we expect that for a sufficiently large R there will exist a region of z such that

$$\frac{zu^*}{\nu} \text{ finite}, \qquad \frac{(h-z)u^*}{\nu} \text{ very large} \quad .$$

Then approximately

$$\bar{v}_x(z) = u^* f\left(\frac{u^* z}{\nu}\right) \quad .$$

At the wall $v'_z = 0$. Then Eq. (8) tells us

$$\nu \left.\frac{d\bar{v}_x}{dz}\right|_{z=0} = \tau_o \quad .$$

Then, as the first term in a Taylor series expansion, we have

$$\bar{v}_x(z) = \frac{u^{*2}}{\nu} z = u^* \left(\frac{u^* z}{\nu}\right) ;$$

i.e., $$f(y) = y \quad .$$

It may be noted that using the other N-S equations, one can show that the next term in the expansion is y^4. Presumably our approximation is good out to some z of the order of $z^* = \frac{\nu}{u^*}$; i.e., the linear law holds out to a $z = \alpha \frac{\nu}{u^*}$. From experiment one finds $\alpha \sim 5$ (this region $0 \leq z \leq \alpha \frac{\nu}{u^*}$ is called the viscous sublayer).

When we are far out of the viscous sublayer but still such that $(h-z) \frac{u^*}{\nu}$ is effectively ∞ , we can also get an expression for f. In such regions the viscous terms are negligible compared to the Reynolds stresses. Then $\frac{d\bar{v}_x}{dz}$ can only depend on z and τ_o. Therefore

$$\frac{d\bar{v}_x}{dz} = \frac{1}{\kappa} \frac{u^*}{z}$$

where κ (the von Karman constant) is presumably universal. Integration yields

$$\bar{v}_x(z) = \frac{1}{\kappa} u^* \ln \left(\frac{z u^*}{\beta \nu}\right)$$

where β is a new universal integration constant. Again from experiment one finds $\kappa \approx 0.4$, $\beta \sim 0.1$. What are the limits of validity? Clearly there is a lower limit $z = (\text{const})\ \nu/u^*$. Empirically, the constant is of order 30. We then have the following picture. For $z \lesssim 5\nu/u^*$ we have a linear law. For $z \gtrsim 30\ \nu/u^*$ there is a logarithmic law. In between there is some buffer region where the two must match up. The upper

limit of the logarithm law clearly occurs when effects of the other boundary become important.

A principal object of theoretical research in turbulence for the last few decades has been isotropic homogeneous turbulence. There is a basic assumption that essential features of turbulence are universal. This is in agreement with our above qualitative description of the cascade process: as energy goes to higher wave numbers, memory of the initial source is lost. Further, since only ensemble averages are eventually desired, a strict probabilistic description is used. For simplicity, the assumption is made that the problem is homogeneous and (usually) isotropic. Mathematically the formulation is:[4] Given an infinite fluid described by Eqs. (1) and (2), and given the initial value of the velocity as a random function of position (described by certain probability laws which are independent of position and direction), we are to determine the probability laws that describe the subsequent motion. An article of faith is included here. It is hoped (conjectured) that nonlinear systems with a large number of degrees of freedom will approach a statistical state which is essentially independent of the initial conditions.

Clearly there is no exact realization of isotropic homogeneous turbulence in nature. Indeed some features realizable in shear flows are certainly absent in the model. However, there are experiments in wind tunnels (see below) in which the turbulence generated does appear to be approximately homogeneous and isotropic. From our point of view, homogeneous isotropic turbulence should probably be considered as an interesting model from which one hopes to get theoretical insight.

B. Status of Experiments

Experimental studies of turbulence (intensities, spectra, correlations, etc.) have been carried out for about the last 50 years with

electronic instrumentation. Although many of these experiments were designed to study flows with complicated geometries that cannot be readily simulated numerically, some of them provide information that can be used to check the validity of the simulations. Unfortunately most experimentalists have attempted to obtain as large Reynolds number as possible in order to study the asymptotic state of the flow, while simulations appear to be possible, at present, only at low and moderate Reynolds numbers. Thus only a small subset of the existing turbulence data is useful for comparison with simulation results.

The measurements referred to below are almost exclusively Eulerian. Due to the obvious experimental difficulties, almost no Lagrangian measurements have been made except for the simplest diffusion experiments. One of the advantages of simulations is the ease with which Lagrangian quantities--multiple particle correlation coefficients, for example--can be calculated.

1. Homogeneous--Isotropic Turbulence

The simplest flow to simulate--unbounded homogeneous, isotropic turbulence--is, unfortunately, impossible to realize exactly in an experiment. There have been many attempts to approximate isotropic turbulence experimentally. Usually these involve using grids in wind tunnels, although measurements have been made on the axis of a circular free jet and measurements of the small-scale structure have been made in tidal channels and in the atmospheric boundary layer. The geophysical flows do not produce data suitable for comparison with simulations because R_λ is large (10^3 to 10^4) in these flows. (Note: $R \sim 2R_\lambda^2$.)

The wind tunnel and jet flows provide more useful data, for comparison purposes, with R_λ in the range of 10 to 200. The measurements generally consist of turbulence intensity, spectra, and second

and third order velocity correlations. Most of the measurements made, up to about 1960, are summarized by Batchelor[4] and Hinze.[5]

In recent years, further experiments have been performed in which great care was taken to make the flow as isotropic as possible. A few examples of these are described below.

(1) Champagne, Harris, and Corrsin[6] carried out an experiment on grid-generated turbulence in a wind tunnel at $R_\lambda = 130$. They measured turbulence intensities, Reynolds stress, the spectra of each of the velocity components, the Taylor microscale, the cross-spectrum of u and v and various second order space-time correlations of u.

(2) Wyngard and Tenekes[7] measured the probability density, skewness and kurtosis of $(\partial v'/\partial t)$ and $(\partial^2 v'/\partial t^2)$ as well as the spectrum of $(\partial^2 v'/\partial t^2)$ at $R_\lambda = 200$. This was in a mixing layer.

(3) Comte-Bellot and Corrsin[8] used grid-generated turbulence and measured turbulent energy, the dissipation rate and second-order correlation functions of velocity at $R_\lambda = 35$ to 70.

(4) Kuo and Corrsin,[9] again using grid turbulence, measured the probability density of v', $(\partial v'/\partial t)$, $(\partial^2 v'/\partial t^2)$, u^2, $(\partial v'/\partial t)^2$ and $(\partial^2 v'/\partial t^2)$ and the kurtosis of $(\partial v'/\partial t)$ and $(\partial^2 v'/\partial t^2)$. These measurements were made at $R_\lambda = 72$ and 830.

These measurements, as well as other recent work and the older studies, provide a wealth of experimental information on "nearly isotropic" flow which can be compared with the results of simulations.

2. Pipe and Channel Flow

The classic experiments are those of Laufer[10] at R = 50,000 and 500,000 in a circular pipe and in a wide channel. The mean profiles, turbulence intensity, the Reynolds stress, turbulent energy, and turbulent-energy-dissipation rate were measured at various distances from the wall down to $zu^*/\nu \approx 3$. The energy spectra of the three velocity

components were measured at several distances from the wall. Some third-order correlations at a point and some second-order spatial correlations were also measured.

Since Laufer's experiments, other workers have continued to investigate these flows. A sample of some of these follow:

(1) Bakewell and Lumley[11] measured the mean profile for $zu^*/\nu > 2$; the energy of the stream-wise component of the turbulent velocity, the spectra and the second-order correlation of u, all for $1 \lesssim zu^*/\nu \gtrsim 40$. These measurements were made at R = 8700.

(2) Wallace, Eckelmann, and Brodkey[12] have studied channel flow at R = 7150. The mean profile and Reynolds stress were measured in the range $1 \lesssim zu^*/\nu > 200$, as well as the second-order correlation of u at $zu^*/\nu = 15$.

These experiments should prove the most valuable for comparison with simulations because these flows can be simulated "exactly."

There is one other flow, Couette flow between rotating cylinders, that can also be "exactly" simulated. However, there does not seem to be any quantitative study of turbulent Couette flow, possibly because of the difficulties of inserting a probe that does not disturb the flow too much. It is possible that laser-velocimeter techniques could be applied to make these measurements.

3. Boundary Layers, Wakes, and Jets

There is an enormous literature (see Coles and Hirst,[13] and Hinze[5] and Townsend [14] for summaries) containing measurements of transition, mean profiles, Reynolds stresses, correlation functions, etc., for boundary layers, wakes, and jets. Because of the difficulty in simulating these flows, these experiments are probably less valuable for comparison purposes.

III TURBULENCE CALCULATIONS

A. Present Status of Turbulence Simulations

Turbulence calculations, using closure approximations of various types, have been made for many years. By suitable adjustments of the turbulence model and the arbitrary constants contained therein, reasonable agreement with experiment has been obtained for particular flows. None of these models appears to be universal and it is not even known whether or not a universal model is possible.

A few direct simulations of turbulence have been carried out in recent years. These were carried out at low and moderate values of R_λ and it is believed that, in these calculations, the resolution was fine enough to resolve all scales down to the Kolmogoroff microscale. The published simulations have treated homogeneous-isotropic turbulence in two or three dimensions.

Lilly [15] and Deem and Zabusky[16] have simulated two-dimensional isotropic turbulence at Reynolds numbers ranging from several hundred to a few thousand (R_λ = 25 to 150). Doubts have been raised about the adequacy of the resolution of the simulations at the higher values of R; however there is no doubt that the resolution was adequate for $R \lesssim 500$ ($R_\lambda \lesssim 60$).

A single set of simulations of three-dimensional isotropic turbulence has been published by Orszag[17] and Patterson,[18] using both finite difference and spectral techniques. In these calculations R_λ = 20 to 40.

To date the results of these calculations have been used to test various theories of turbulence in both two and three dimensions. These

calculations are extremely valuable as tests of theories of isotropic turbulence because homogeneous isotropic turbulence cannot be exactly realized in the laboratory or in geophysical flows. In particular, the initial conditions of the simulations can be varied arbitrarily and any desired information about the flow can be obtained from the simulations with relative ease.

Finally, it should be noted that a number of workers are carrying out simulations that are, as yet, unpublished.

B. Calculations of Interest

It should be emphasized that a direct numerical attack--i.e., starting from Eqs. (1) and (2)--on most problems of practical interest is impossible now and indeed may always be so. The reasons are the complicated geometrics involved and, more importantly, the range of scales that must be described. Below we indicate that in three dimensions the number of computations to be done for one time-dependent problem is of order $R^3 \ell nR$. Practically, R can be 10^6 to 10^7 or larger. The goal of numerical simulation is to determine universal features of turbulence. If such are found, one hopes to develop approximate equations for the average flow such that the fine sides need not be resolved. (Even a negative result--that there are some features of different flows that are not universal--would be important.)

Let us consider then what problems might be usefully attacked numerically with the new generation of computers. We choose two criteria.

1. Benchmark Problems

These are problems on which approximation methods could be tested. They should be such that as few assumptions as possible are put into the calculation. Existing (or possible) experimental results

should be available. Flows in pipes and channels of both the Poiseuille- and Couette-type are of this nature. So is Couette flow in concentric cylinders. Boundary-layer flows over plates are less useful since some extra conjectures as to the behavior at infinity are necessary.

2. Theoretical Problems

These are problems whose solution may lead to theoretical insight--and then to practically useful approximations. (Of course, they can also be used to test existing approximations.) Problems of this type are:

(1) Studies of flows just above the critical Reynolds number: These could greatly improve our understanding of the mechanics of transition. An example is the transition in Poiseuille flow in pipes. Linear-stability theory[19] says this is stable for all R. In practice the transition occurs at $R \sim 2200$. Clearly the Landau picture[1] of transition discussed above cannot be correct here. The general theory of nonlinear instabilities could be much clarified.

(2) Problems of isotropic homogeneous turbulence: Presumably these should be done with periodic boundary conditions. While idealized these can be considered as numerical experiments. For example, if one has results for a very fine grid, one could see how well this could be modeled using a coarser grid with approximate boundary conditions (in wave number space).

(3) Two-dimensional problems: Though also somewhat ideal, there is some indication that the atmospheric problem is of this type. (There is some disagreement as to whether "turbulence" can be two-dimensional since one method of vorticity production--vortex stretching--is absent. However, adopting the view that turbulence refers to hydrodynamical problems that require a statistical description, one concludes that there is such a thing.) Theoretically the two-dimensional problems are particularly interesting since, as we shall see, problems with much more interesting Reynolds numbers can be computed than are possible in three dimensions.

Indeed, there are indications that macroscopic properties of such flows are insensitive to the Reynolds number.[20] However, these simulations may not have resolved properly all necessary scales of motion. For accurate simulation the entropy-dissipation spectrum should lie well within the resolvable scales.

(4) Varying boundary conditions: The effects of boundaries can be studied by numerical experiments. Instead of no-slip conditions in pipes, one could consider other conditions to see what the effects are. (These are "experiments" that are not feasible in laboratories.)

Of course, in order to be useful, the numerical calculation must be done for problems of sufficiently high Reynolds number. For a given problem, we must be above the critical value. Also, for any material increase in our knowledge, R should be significantly greater than has been used for calculations before.

Some very interesting questions seem beyond the range of present numerical attack. One such is the existence of an equilibrium range and in particular an inertial subrange where the Kolmogoroff spectrum holds. Theoretically this spectrum is a little peculiar. We see that Eq. (7) related E(k), (the energy density), to $\epsilon^{2/3}$ (the energy dissipation rate). Since ϵ is itself a fluctuating quantity, what is meant here? Presumably an average is meant here. But E(k) is also an average. It seems strange to relate one average to a fractional power of another. Experimentally there is also some question. Kolmogoroff's assumption[2] implies not only the 5/3 spectrum but also that other statistical quantities of the small scales have universal values when put in non-dimensional form with ϵ and ν. In particular, in the inertial range there should be universal scaling with ϵ alone. Thus the skewness

$$\frac{\overline{|\Delta \underset{\sim}{v}'(r)|^3}}{\left(\overline{|\Delta \underset{\sim}{v}'(r)|^2}\right)^{3/2}}\ ,\ \text{and the kurtosis}\quad \frac{\overline{|\Delta v'(r)|^4}}{\left(\overline{|\Delta v(r)|^2}\right)^2}\ ,$$

where $\Delta \underset{\sim}{v}'(r) = \underset{\sim}{v}'(\underset{\sim}{x} + \underset{\sim}{r}) - v'(\underset{\sim}{x})$, should be universal constants. However, measurements suggest that skewness and kurtosis rise with R.

Attempts to accommodate these difficulties give rise to slight modifications of the 5/3 power law.[21,22] Also, models[23] and approximate calculations tend to give rise to other laws which are similar to this but not the same.

Can we resolve the question by numerical simulation? Probably today we cannot. Thus, for the existence of an inertial range, we saw that one must have a range of k such that simultaneously

$$kn \ll 1 \text{ and } k\ell \gg 1 \ .$$

Of course it is somewhat arbitrary as to what these inequalities must be, but let us assume 100:1 is a reasonable value. To see a power law one would need several decades in k. Let us suppose this is two. At the lower end (k_o) we must have

$$k_o \eta \approx 10^{-4} \ , \quad k_o \ell \sim 10^2$$

and at the upper end (k_1) we have

$$k_1 \eta \simeq 10^{-2} \ , \quad k_1 \ell = 10^4 \ ;$$

therefore

$$\ell/\eta \sim 10^6 \ .$$

Indeed, as we see below, L/η is the order of the number of linear grid points one needs for a calculation, and 10^6 will be found to be prohibitive.

The Reynolds numbers that can be used for a given computer are determined by the resolution required. In the case of isotropic homogeneous turbulence, we have only two lengths--L, a periodicity length, and η, the Kolmogoroff length scale. Somewhat more stringent conditions are met in contained flows, such as pipes. Here, in addition to r_o (the pipe radius) and η, we have the thickness of the viscous sublayer $\delta_v \sim 5\,\nu/U^*$, and the thickness of the region over which there is a change from the linear law to the logarithmic region, $\delta_b \sim 30\,\nu/U^*$. From experiment we take U^* (the friction velocity) and U the turbulence level to be $\sim \frac{U_m}{20}$ with U_m the mean velocity flow. With the Reynolds number R defined by $R = U_m r_o/\nu$, the various lengths are related as:

$$\frac{\eta}{r_o} \sim \frac{10}{R^{3/4}} \quad , \quad \delta_v \sim \frac{100}{R} \quad , \quad \delta_b \sim \frac{500}{R} \quad .$$

Table 1 gives these rates for various interesting Reynolds numbers.

Table 1

RATES FOR SELECTED REYNOLDS NUMBERS

R	η/r_o	δ_b/r_o	δ_v/r_o
2,000	0.03	0.25	0.05
5,000	0.017	0.10	0.02
10,000	0.010	0.05	0.01
20,000	0.006	0.025	0.005
40,000	0.004	0.125	0.0025
100,000	0.002	0.005	0.001
1,000,000	0.0003	0.0005	0.0001

From this table some conclusions can be drawn. For $R \leq 10^4$, adequate resolution is achieved with a linear grid of about 100 points; $R \sim 2 \times 10^4$ to 4×10^4 requires about 200 points; $R \sim 10^5$ requires about 500 to 1000 points, while $R \sim 10^6$ requires between 3000 and 10,000 points.

IV COMPUTATIONAL METHODS

The general problem is to find solutions of Eqs. (1) and (2) in a region V with appropriate boundary conditions on the bounding surface S. Many schemes have been proposed (and used) for numerical calculation. They can be characterized in many different ways.

One way is in terms of Eulerian or Lagrangian form. The Lagrangian formulations are characterized by a coordinate system that moves with the flow. For flows that are relatively uncomplicated, this is a very attractive method. For the violent flows of a turbulent fluid there are serious disadvantages and some possible advantages. The advantage might be that the flow would look much simpler in Lagrangian coordinates. The disadvantage is that an initially well ordered mesh will become badly distorted. The problem then is to keep track of which particles are in the immediate vicinity of a given particle in order to determine the forces they exert. While work has been (and is being) done on this formulation, it is felt that the means to implement such a calculation efficiently are not sufficiently well understood to be used in evaluating feasibility of turbulence calculation here (i.e., the study of such methods is a worthy research project in its own right). Accordingly, we restrict ourselves here to the N-S equations in the form given above. (A recently proposed method of Chorin[24] which is somewhat related may be noted. In two dimensions the method follows the motion of a finite number of vortex elements; however, the accuracy obtainable and the extension to three dimensions is not yet known.)

Another way of characterizing computational methods is according to whether they are finite difference in physical space, or spectral methods, or a combination.

The difference methods can be subdivided into whether they calculate the "primitive" variables P and $\underset{\sim}{v}$ or related ones. (In two dimensions these are a stream function and a vorticity function. In three dimensions one needs a vector potential.) For two-dimensional calculations the use of those derived variables has been rather popular. However, there is some suspicion that more accurate results are obtainable using the primitive variables. The reason is that in calculating the vorticity function one is trying to find the derivative of a rapidly varying quantity. Accordingly we will make estimates assuming P and $\underset{\sim}{v}$ are being calculated directly.

In assessing finite difference methods in turbulence calculations it is important to note that turbulence is characterized by a wide range of excited scales of motion. (See Section III for an example.) To deal with this, one can introduce very many grid points. Another approach, when one knows a priori where the small scale motions will be, is to appropriately stretch the coordinate scale. For example in the pipe flow (similar to the flow described in Section II), most of the turbulent production occurs at a distance from the walls that is only a few percent of the radius. Accordingly, one would like to arrange the mesh so that a significant fraction (perhaps one-fourth) is devoted to that small region. Actually one will probably want to go in both directions.

Another procedure[20] is to go to higher order difference schemes. Thus one can apparently obtain the same accuracy (in d dimensions) by using $(\frac{1}{2})^d$ of the mesh points needed in a second-order difference calculation by going to a fourth-order scheme. This can be very significant. In a two- or three-dimensional calculation, one has to store at any time step $2N^2$ or $4N^3$ numbers, respectively. Since the N's of interest are on the order of 128 to 1024, a factor of four (or eight) is of considerable importance in terms of memory requirements. One, of course,

pays a price--more arithmetical operations are needed and hence computing time is increased. Estimates of what one gains and loses by going in this direction are very similar to the comparison given below between pure spectral methods and a mixed spectral-difference approach.

Spectral methods[20] involve expanding the fields in terms of an appropriate finite set of orthogonal functions. The N-S equations then become coupled ordinary differential equations (in time) for the expansion coefficients. This formulation has a number of attractive features. Thus, for a given order of accuracy, one needs to consider many fewer degrees of freedom. Also convergence, when it occurs, is very rapid (i.e., to pass from say 5% accuracy to 1% seems to require very few additional orthogonal functions). The disadvantages are that one must find functions with appropriate properties corresponding to a given geometrical arrangement. Further the transforms that the expansion implies must be evaluatable by some efficient algorithm, such as the fast Fourier transform. However, such functions are known for the simple problems envisaged here. For periodic conditions one can use Fourier transforms. For pipes and channels, one can use Chebyshev polynomials. Another disadvantage is an increase in computing time. (We will come back to this.)

Actually, one is not constrained to a pure form of finite difference methods or to pure spectral methods. A mix in which some coordinates are treated one way and others another is possible, and probably even optimal.

To get some possible estimates of computer requirements, we consider two methods of calculation.[26] Probably neither is optimal (in particular, cf. Section V), but they do describe the kind of ranges one might go over if one tried various of the alternatives sketched above.

A. Method 1

One solves the velocity equation as a marching problem with finite differences in space and time (assuming a second-order scheme). Assuming N mesh points in each spatial direction, in two (three) dimensions there are $3N^2(4N^3)$ variables to be obtained at each time point. If N is large the main computation problem is to determine the pressure from the Poisson equation

$$\nabla^2 p = -\nabla \cdot (\underset{\sim}{v} \cdot \nabla)\underset{\sim}{v} \quad . \tag{3}$$

Let us suppose that the solution can be obtained by a method equivalent to a Fourier transform technique. (With periodic boundary conditions, this would indeed be a Fourier transform. For pipes and channels this might be a Fourier transform in two dimensions and Chebyshev expansion in the third.) We remark that this is not necessarily the most efficient method of solution. With, for example, transforms in all but one dimension and treating the remaining dimension by methods of inverting tridiagonal matrices or Hockney's method[26] one can get by with slightly fewer operations. (The saving though is only a factor on the order of two or so, and hence we ignore it.) For large N, then our major computing effort is to take the Fourier transform of the pressure equation, and then invert. This involves two Fourier transforms on $N^2(N^3)$ variables. Using the fast Fourier transform algorithm we then need $2\{N^2 \ln_2 N^2$ additions plus $\frac{1}{2}N^2 \ln_2 N^2$ multiplications$\}$ or $2\{N^3 \ln_2 N^3$ additions plus $\frac{1}{2}N^3 \ln_2 N^3\}$. For simplicity let us assume one multiplication to be equivalent to two additions. (The times for these is indeed roughly that of the Illiac.)

Then the additions needed in 2D are

$$4N^2 \ln_2 N^2 \quad , \tag{4a}$$

and in 3D we need

$$4N^3 \ln N^3 \tag{4b}$$

for one time step.

Actually these are only asymptotic formulas. For the range of N we will be considering they are not too bad for two-dimensional estimates, but rather poor for the three-dimensional case. Indeed we have completely omitted the calculation of the velocity terms in real space and the Fourier transform of the pressure in transform space. Since these are both local, in that at most a point and its close neighbors are involved, this gives a number of operations $\sim N^2$ (N^3). A more correct estimate of the number of equivalent additions is then in 2D:

$$\mathcal{N}^N \cong 4N^2 \ln_2 N^2 + 50N^2 \qquad \text{per time step} \tag{5a}$$

and in 3D:

$$\mathcal{N}^N = 4N^3 \ln_2 N^3 + 150N^3 \qquad \text{per time step} \tag{5b}$$

How many time steps are needed? For stability we need*

$$\Delta t < \frac{\Delta x}{U} \approx \frac{L}{NU}$$

where U is typical of the overall flow velocity and L is a typical dimension of the system. The total time T that we might want to calculate is some significant multiple (perhaps about ten) of the macroscopic time L/U for something to happen to the whole system. The number

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*Note: The fact that we may need a fine grid resolution near a wall may still not cause trouble with the Courant stability condition. Thus in the boundary layer the flow is nearly parallel to the wall and the condition involves only the projection of the convecting velocity on the grid increment in that direction.

of time steps N_T then is such that

$$N_T \Delta t \simeq mL/U \qquad \text{or} \qquad N_T \approx mN \quad .$$

Therefore the total number of operations is

$$\text{in 2D} = mN^3\{4\ln_2 N^2 + 50\} \approx N^3\{40\ln_2 N^2 + 500\} \tag{6a}$$

$$\text{and in 3D} = mN^4\{4\ln_2 N^3 + 150\} \approx N^4\{40\ln_2 N^3 + 1500\} \tag{6b}$$

The total time for such a calculation is then the above numbers times τ, the addition time.

B. Method 2

Here one imagines using purely spectral methods. For example, suppose we can use Fourier transforms. If k is the maximum wave vector in each direction we have

in two dimensions	$2 \cdot (2k)^2$	real variables
and in three dimensions	$3 \cdot (2k)^3$	real variables .

Here the whole problem is involved with calculation of the nonlinear terms in the velocity equation (which were asymptotically unimportant in Method 1). An efficient means of doing this is by passing back and forth between Fourier space and real space. It has been shown[27] that in two (three) dimensions this can be done with 20 (72) fast Fourier transforms. Therefore, for one time step the number of effective additions is

$$\mathcal{N}_a^k \sim 40(2k)^2 \ln_2 (2k)^2 \qquad \text{in two dimensions}$$

and

$$\mathcal{N}_a^k \approx 144(2k)^3 \ln_2 (2k)^3 \qquad \text{in three dimensions.}$$

of time steps N_T then is such that

$$N_T \Delta t \simeq mL/U \qquad \text{or} \qquad N_T \approx mN \quad .$$

Therefore the total number of operations is

$$\text{in 2D} = mN^3\{41n_2N^2 + 50\} \approx N^3\{401n_2N^2 + 500\} \tag{6a}$$

$$\text{and in 3D} = mN^4\{41n_2N^3 + 150\} \approx N^4\{401n_2N^3 + 1500\} \tag{6b}$$

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and

$$\mathcal{N}_a^k \approx 144(2k)^3 1n_2(2k)^3 \qquad \text{in three dimensions.}$$

For comparison, it is important to note that the accuracy for a given k is that by second order finite differences for an equivalent mesh of size

$$N_e = 4k \quad .$$

Expressing everything in terms of N_e we see that the spectral method requires

$\frac{1}{2}N_e^2$ variables in two dimensions

and $\frac{3}{8}N_e^3$ variables in three dimensions.

This can be compared with the numbers $2N^2$ and $4N^3$ for Method 1 (i.e., in two dimensions we have a saving in memory by a factor of four and in three dimensions by more than a factor of 10).

What do we pay in computing time? Using the asymptotic formulas we obtain

in two dimensions $$\frac{\mathcal{N}_a^k}{\mathcal{N}_a^N} = 2.5 \quad ,$$

and in three dimensions $$\frac{\mathcal{N}_a^k}{\mathcal{N}_a^N} = 4.5 \quad .$$

Actually the time disadvantage here is quite overestimated. The relatively small N that would be of interest is such that these ratios are more like a factor between one and two. The main disadvantage of the spectral methods seems more likely to be their lack of generality and increased complexity of coding.

V ILLIAC IV CHARACTERISTICS

To estimate computing times it is useful to have a brief description of the properties of Illiac IV, as perhaps it may be within a year. (Only characteristics essential for our evaluation are given.) The essential conclusion is that memory requirements drastically limit the usefulness of the Illiac in turbulence calculations. The essential characteristics include:

(1) Processors. There are 64 parallel processors. These can all be performing the same operations, or some can be inoperative. One PE can do a 64-bit floating-point add in 30 ns and a floating-point multiply in 600 ns.

(2) Memory.

(a) There is a 1.28×10^5 64-bit-word working-array storage.

(b) There is a 16×10^6 64-bit-word disk storage. The transfer rate from disk to array is 10^7-word per second. This is somewhat misleading, so we spell out the details. The disk is arranged in 52 bands, each with 300 pages of 10^3 64-bit words. The unit of transfer is one page. Then 133 μs are needed to transfer a page. However, this requires having immediate access to the desired page. The disk rotates in 40 ms. If one has to seek out a random page we might expect the time to be ~10 to 20 ms. Further limitations are: (i) it takes 133 μs to switch between bands, and (ii) one cannot read on one band and write on another simultaneously. Indeed it takes another 133 μs to change from reading to rewrite. The principal conclusion here is that to use the 10^7-words-per-second transfer rate one must have stored data in exactly the right place. As much as possible all numbers needed during a set of calculations should have come from a given page and neighboring pages.

(c) There is also a buffer disk of about ten times the Illiac disk and a transfer rate about one-tenth that of the Illiac disk.

(d) There is a Unicon laser storage device with an on-line storage capacity of $\sim 10^9$ 64-bit words and infinite off-line storage with a transfer rate of 10^5-words-per-second.

Properties (c) and (d) have little effect on computing capability but do restrict the amount of information one might want to bring out of the computer.

It is somewhat instructive to compare the Illiac IV with other machines. Take as a basis the roughly comparable IBM 360/195 and the CDC 7600. Their characteristics are approximately:

- Add time ~ 100 ns
- Multiply time ~ 200 ns
- Fast Memory $(0.5 \text{ to } 1) \times 10^6$ 64-bit words.

We note that since the arithmetic times are of the order one-third that of the Illiac, the factor of 64 (due to the many processors) is effectively reduced to a factor 20. There is, however, the question of the efficiency with which one can employ the 64-fold parallelism. For example, suppose one were calculating using a $(128)^3 \sim 10^6$ grid. Let us suppose that the bulk of the calculation can be done with the full 64-fold parallelism. The boundary points are of order $6 \times (128)^2$ ~ 60,000 points. These are about 10% of the calculation and might require only six processors. This will then double the computing time. A not unreasonable estimate of the Illiac advantage over the IBM 360/195 or CDC 7600 is then about ten. (This assumes the whole problem be contained in the array memory: If not, for the reasons given above, this advantage may be drastically reduced.)

Similar estimates for Star and ASC are more difficult since much less is known about how well these will work. Guesses as to the improvement of Star over the CDC 7600 seem to be a factor of three while the ASC improvement might approximate two (for one pipe) to

eight for four pipes. Again the problem of implementing the parallelism remains a question. (One interesting property of the ASC is the apparent capability of increasing the fast memory to (4 to 16) $\times 10^6$ words. This could be a great advantage for turbulence calculations.)

Let us now try to make some estimates of computing times for problems of various sizes.

An absolute minimum is obtained by taking the number of equivalent additions, multiplying by the add time and assuming we can use the full 64-parallelism perfectly. In Tables 2 and 3 we give the results in two and three dimensions corresponding to Methods 1 and 2. We give number of variables, time per step, and total time (assuming $T = 10N\Delta t$). (Here t_1, t_2 are time per step for Methods 1 and 2; U_1, U_2 the number of variables; and T_1, T_2 are the corresponding total times.) An important point to be noted from these tables is that for comparable accuracy t_2 is not much greater than t_1, while the number of variables to be kept is much smaller.

To be realistic we should allow some factor for inefficiency. Accordingly, all times should really be multiplied by some factor--probably between two and ten: Let us say five. What problems are now within reasonable limits? Suppose we put an upper limit of 100 hours for one problem. Then we see that in two dimensions we can at most consider $N = 2048$ (or $K = 512$), while in three dimensions the limits are $N = 256$ ($K = 64$). If we drop our limit to approximately ten hours the numbers are $N = 1024$ ($K = 256$) in two dimensions and $N = 128$ ($K = 32$) in three dimensions. We note that while these estimates are quite rough, it is clear that the rapid rise in total computing time with increasing N suggests these upper limits cannot be wrong by more than a factor of two.

Table 2

CALCULATIONAL TIMES FOR TWO DIMENSIONS
(All Times in Seconds)

N	K	t_1	t_2	V_1	V_2	$T_{1/10}$,	$T_{2/10}$
64	16	2×10^{-3}	2×10^{-3}	8×10^{-13}	2×10^{3}	1.2×10^{-1}	1.3×10^{-1}
128	32	8×10^{-3}	9×10^{-3}	3.2×10^{-14}	8×10^{3}	1.0	1.0
256	64	3.5×10^{-2}	4×10^{-2}	1.3×10^{5}	3.2×10^{4}	8.9	1.0×10^{1}
512	128	1.5×10^{-1}	2×10^{-1}	5.2×10^{5}	1.3×10^{5}	7.7×10^{1}	1.0×10^{2}
1024	256	6×10^{-1}	8×10^{-1}	2×10^{6}	5.2×10^{5}	6.1×10^{2}	8.2×10^{2}
2048	512	2.7	3.8	8×10^{6}	2×10^{6}	5.5×10^{3}	7.8×10^{3}
4096	1024	11.8	16.7	3.5×10^{7}	8×10^{6}	4.8×10^{4}	6.8×10^{4}

Table 3

CALCULATIONAL TIMES FOR THREE DIMENSIONS
(All Times in Seconds)

N	K	t_1	t_2	V_1	V_2	$T_{1/10}$	$T_{2/10}$
16	4	3.8×10^{-3}	3.1×10^{-3}	1.6×10^{4}	1.5×10^{3}	6×10^{-2}	5×10^{-2}
32	8	3.2×10^{-2}	3.3×10^{-2}	1.3×10^{5}	1.2×10^{4}	1.0	1.0
64	16	2.7×10^{-1}	3.3×10^{-1}	1.0×10^{6}	10^{5}	2.4×10^{1}	2.1×10
128	32	2.3	3.1	8.2×10^{6}	7.8×10^{5}	3.0×10^{2}	4.0×10^{2}
256	64	2.0×10^{1}	3.0×10^{1}	6.5×10^{7}	6.3×10^{6}	5.1×10^{3}	7.6×10^{3}
512	128	1.6×10^{2}	2.6×10^{2}	5.2×10^{8}	4.8×10^{7}	8.2×10^{4}	1.3×10^{5}

However, in considering using Illiac IV for such turbulence calculations one should also consider the relatively small array memory. We describe the three-dimensional case in the following paragraph.

Suppose one restricts attention to problems that can be done using only the fast array memory. Presumably at any time we need to be storing ~2V words. For Method 1 we are restricted to $N < 32$ and for Method 2 to $K < 16$. Clearly, in order to solve interesting problems, use will have to be made of the Illiac disk. The problems that are then possible are those with accuracy comparable to that achievable with second-order finite differences with $128 \leq N \leq 256$. Methods like Type 1 are clearly limited by disk space to $N \approx 128$. Here the disk is completely filled (indeed slightly over) by numbers one needs to be keeping. The situation is similar in methods of Type 2 for $K = 64$. In any case one is faced with a situation where, at any instance, almost all the numbers are being stored in the disk. Then the disk transfer rate also gives some time limitations.

Thus, consider a situation such as Method 1 with $N \approx 128$. (We say approximately since the exact case does not fit in; however, we imagine this taken care of, for example, by going to 32-bit words.) At any time step we are storing $\sim 8 \times 10^6$ words from a previous time step. The remaining 8×10^6 memory places are to be repeatedly used during the time step. If coding could be done perfectly, one could imagine proceeding so: The 6×10^6 velocity variables from a given step are fed in. From those the terms $v \cdot \nabla v$, $\nabla^2 v$ and $\nabla \cdot (v \cdot \nabla v)$ are computed. Since these are so numerous they must be taken out to the disk. Fourier transforms are then taken plane by plane--again they must be taken out. Next one transforms in the direction perpendicular to the planes--this is again a pass in and out. The same must be done to invert the transform of the solution of the Poisson equation. Finally, all terms must be combined to obtain the next values of the

velocities. Thus there are a number of passes (about ten) in and out of the disk. (Not all the passes require a full 8×10^6 words.) However, the net effect is that something like $5 \times (8 \times 10^6)$ words have to go in and out of the disk. With a transfer rate of 10^7 words per second, this is on the order of six seconds. We remark that with our previous estimate of (5×2.3) seconds for computing time per step, this does not materially change the time that the total calculation would take. It does indicate that just enlarging the disk manifold does not change the situation as to what problems can be attacked.

The real limitation imposed here by the disk is hidden by our assumption of perfect coding. If at any given pass all the numbers are not on the same page with all those needed to compute the next stage from them, and indeed all pages are not arranged to be accessible in sequential order, our assumed transfer of rate of 10^3 words in 133 μs drops to 10 ms or less. Clearly, to perform calculations where a large fraction of the disk must be repeatedly read and rewritten, we require a very careful choice of method of calculation and great care in coding. It would seem that the main determinant for the method of calculation may well be the ease with which it permits coding for efficient access and egress from the disk. This is a difficult problem and will take careful study. However, we believe it not necessarily insoluble. (In Appendix B, a possible, though probably not optimal, approach is sketched.)

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VI CONCLUSIONS

In principle the advent of a new generation of computers (in particular the Illiac IV) makes possible the numerical solution of some interesting problems in turbulence. By themselves, however, they will not give the ideal goal, an ability to predict the flow for a wide variety of practical problems. Many basic questions and problems involve Reynolds numbers far beyond present capabilities. What can be achieved are:

(1) The obtaining of solutions to a number of benchmark problems. These can be used to test approximation methods for Reynolds numbers $R < 10^4 - 4 \times 10^4$.

(2) The obtaining of greater insight into the nature of turbulence. This would be the result of a number of numerical experiments.

Such an effort seems desirable; probably it does not require a massive program.

Doing these problems on the Illiac IV presents severe difficulties. The array memory is small compared to the number of independent variables needed, at least for some of the more obvious methods of computation. If desired, a reasonable program might be the following:

(1) An attempt should be made to see how a typical problem could be coded so as to achieve a resolution of distances of the order of 1/128 of the overall scale (e.g., in a pipe one wants to resolve distances ~ 1/128 of r_o). The method of calculation used will be primarily determined by the size of the array memory. One approach is to see how one can program so as to efficiently access the disk memory. If it is found that this cannot be done well, an alternative might be to go to very high-order difference schemes. Presumably a significant

reduction of the number of independent variables can be achieved, at the expense of an increase in computing time. The exact advantages and disadvantages obtained in this way are not known and would have to be determined.

(2) Assuming this coding problem has been solved, one would calculate the problems sketched in Section III. Since many of the results are desired for theoretical insight, the calculation should be carefully followed by an advisory group. The actual choice of what to calculate next will depend on what has already been found.

Since huge quantities of numbers will be generated, it will be necessary to be rather selective as to what one takes out of the computer. For example, one might want only to read out every 10 to 20 time steps the various quantities for which experimental results are available (see Section II-B). In particular, though, one would want the energy spectrum, vorticity spectrum, skewness, and kurtosis.

(3) Since the amount of experimental information at present far exceeds the amount of numerical results, an experimental program does not now seem in order. When (if?) numerical results are available, there may be some questionable features which would suggest specific experiments.

(4) Even if the problem of appropriate coding for the Illiac IV proves insoluble, some parts of the above program could be usefully done on a lesser computer. For example, even with an IBM 360/195 or a CDC 7600 some of the above problems could be done, but at Reynolds numbers lower than those envisaged for the Illiac.

It is, of course, possible that the computing estimates that have been made are too high, in that alternative methods of calculation may require less computing time. However, it is thought that our estimates are fairly conservative. In any event the primary conclusion--that the computations needed increase dramatically with Reynold's number--is certainly correct.

Appendix A

A NAVIER-STOKES COMPUTER?

1. Summary

The conceptual design of a special-purpose computer array to solve the N-S (Navier-Stokes) equation is discussed below. An array of 100 to 150 special processors (each roughly like a bare modern minicomputer) with a 100-ns add time will permit the solution of the second-order difference approximation to the N-S equation in a total processing time (for 10^4 time steps) of several minutes (N = 64) to seven hours (N = 1024). This processing time could be decreased by paralleling and/or pipelining in each processor. The total hardware cost is of the order of $\$5 \times 10^5$ to 10^6.

The major problem is not the processors but memory. A total of 6×10^7 (N = 64) to 2×10^{11} (N = 1024) bits are required. It appears that for the largest calculation, a cost of no more than $\$10^{-4}$ per bit is required. In order to match memory speed to processor speed, each processor will require 16 K to 32 K of 100-ns memory in at least eight banks and four or eight drums or disks with about an 8×10^{-3}-s average access time, a 5×10^{-6}-s-per-word transfer time and 1 to 2×10^6 total storage. That is, a total of 1.6 to 3.2×10^6 words of 100-ns memory and 400 to 800 independent disks, each with a capacity of 1 to 2×10^6 words is required to handle even the N = 512 problem. The total cost of this memory will certainly be in excess of $\$10^7$.

2. Discussion

Assume that there are K V-Boxes, L F-Boxes, and M P-Boxes, and a memory (all are described in the following sections), then the total time to calculate V and P on all mesh points for one time step is

$$T = \left(\frac{T_V}{K}\right) + \left(\frac{T_F}{L}\right) + \left(\frac{T_P}{M}\right)$$

$$= N^3 \left[\left(\frac{200}{K}\right) + \left(\frac{8\ell n_2 N}{L}\right) + \left(\frac{13}{M}\right)\right] t_o \quad .$$

The total cost, C_T, is the sum of the cost of the V, F, and P boxes, the C box, and memory, and is

$$C_T = KC_V + LC_F + MC_P + C_c + C_M$$

$$= (K + L + M)\,(c_a + c_m + 8c_r + c_w) + C_c + C_M \quad ,$$

where

c_a = Cost of one adder

c_m = Cost of one multiplier

c_r = Cost of one register

c_w = Cost of wiring.

The memory cost, C_M, is

$C_M = 224\, c_o N^3$, or

$$C_M = 64\, c_o n + 32\, c_1 (7N^3 - 2n)$$

$$= 64\, (c_o - c_1) n + 224\, c_1 N^3$$

depending on whether a single high-speed memory, with a cost per bit of c_o or a two-level memory ($2n$ words of fast memory at a cost per bit of c_o and the remainder slow memory at a cost per bit of c_1) is used. The high-speed memory has a cycle time $\tau_o \leq t_o$ and the slow-speed memory has a cycle time of τ_1.

Assume, for the sake of making an estimate, that roughly the same time is spent in each of the boxes. That is,

$$\frac{200}{K} \approx \frac{8 \ell n_2 N}{L} \approx \frac{13}{M} \quad .$$

Taking N = 256, it is seen that M = K/10 and L = K/4 will satisfy. Then,

$$T \approx (600\, N^3/K)\, t_o \quad .$$

Taking N in the range 64 to 1024 and t_o = 100 ns (currently available adders achieve this), it is seen that K = 100 gives T, the total compute time per time step, in the range 10^{-2} to 60 s, for a maximum of, say, seven hours per computation.

If seven hours per computation is too long, it can be considerably shortened, by at least an order of magnitude. If the (second order) difference approximations to the Navier-Stokes equations are examined in detail, it becomes clear that it is possible to build into each of the processors a fairly high degree of parallelism and/or pipelining and thus achieve the speedup. However, there seems to be little point in this because, as will be seen, the memory requirements are already difficult and speeding up the processors will only make them worse.

In order to keep the processors busy it is necessary to supply some memory with an access time τ_o,

$$\tau_o \leq t_o = 100 \text{ ns}.$$

If we had a set of main memories, one for each processor, it is clear that even for an N = 64 calculation something of the order of 2×10^6 words (64×10^6 bits) of memory would be required. If 100-ns memory costs \$0.01 per bit, the cost is of the order of $\$6 \times 10^5$. For N = 1024, the cost is of the order of $\$2 \times 10^9$.

If a number of secondary, slow speed, memories are used, then the access time (see below) could be in the range 1 to 10 μs depending on the number of banks (one to eight) in each of secondary memories. Still, between 6×10^7 and 2×10^{11} bits are required. If the cost per bit were only $\$10^{-4}$ the largest calculations require of the order of $\$20 \times 10^6$ worth of memory.

Going to drums or disks for the secondary memory will help, but not too much. If each primary memory (2n words, 100-μs access time) has ℓ disks, each with an average access time of τ_o and a transfer time per word of τ_1, then

$$\frac{\tau_a}{n} + \frac{\tau_1}{\ell} \lesssim 2 \times 10^{-6} \text{ s} \quad .$$

The right-hand side is an estimate of the time the fastest box (V, F, or P) takes to perform its calculation on one mesh point. Taking

$$\tau_o = 8 \times 10^{-3} \text{ s}$$

$$\tau_1 = 5 \times 10^{-6} \text{ s} \quad .$$

A number of pairs of n and ℓ can be chosen to satisfy this inequality. Some possible choices are

$$n = 8\ K,\quad \ell = 4$$

$$n = 8\ K,\quad \ell = 8$$

$$n = 16\ K,\quad \ell = 4.$$

The first gives a time just a bit too long and the others satisfy the inequality comfortably. Therefore, 100 fast memories (100 ns) either of 16 K or 32 K words capacity and 400 or 800 independent disks, each with 10^6 to 2×10^6 words, are required to handle the N = 1024 problem. If each disk-drive costs only $\$10^4$, the disk cost is \$4 to 8×10^6. Also 5×10^7 to 10^8 bits of fast (100 ns) memory is required. If the cost were only $\$10^{-3}$ per bit the cost is \$5 to 10×10^6.

Finally a word on the processor costs. Each of these boxes appears roughly equivalent to a good minicomputer. The bare cost of the processor should be similar, say \$5 to 10×10^3. This is equivalent to taking

$$c_a \approx \tfrac{1}{2} c_m \approx c_r \approx \$100 \text{ to } \$200$$

and assuming that the wiring cost is two to three times the hardware cost. The total processor cost is then of the order $\$5 \times 10^5$ to 10^6, which is considerably smaller than the memory cost.

3. The V Box

The V box calculated $\vec{V}$ and the right-hand side of the Poisson equation for P on one mesh point. Assume that all of the operations necessary to perform these calculations are performed sequentially and that the adder and multiplier are not pipelined. The total time for this calculation, t_v, is then

$$t_v = \text{(number of addition)(addition time)}$$

$$+ \text{(number of multiplications)(multiplication time)}.$$

These calculations require approximately 100 additions and 50 multiplications. Therefore,

$$t_v = 200\ t_o \quad .$$

The total time to calculate on N^3 points, assuming no parallel V boxes, is

$$T_V = (200\ N^3)\ t_o \quad .$$

In the worst case, two successive operations will require four different operands. To avoid waiting for the second pair of operands to be fetched from memory, they should be fetched while the first operation is being performed. This requires two registers to hold the current operands and two registers to hold the next operands. A result register to hold the result of the current operation is needed as well as a holding register to save the partial sum accumulated in calculating $\vec{V}$. Finally two registers for constants are probably needed. In total:

- Two current operand registers
- Two next operand registers
- Two constant registers
- One result register
- One holding register

for a total of eight registers are needed. In addition, one adder and one multiplier are also required.

The cost of the hardware in a V box, and assembly cost, is

$$C_V = c_a + c_m + 8c_r + c_w$$

where

C_V = Total cost of one V box

c_a = Cost of one adder

c_m = Cost of one multiplier

c_r = Cost of one register

c_w = Cost of wiring up a V box.

4. The F Box

The F box is just a machine to calculate the FFT on N^2 points in a fixed plane and then over all planes. A large number of FFT machines have been built and it is probably more accurate to estimate the time, T_F, to calculate the required FFT's as well as the cost, C_F, of a single FFT machine, by using the speed and cost figures for existing machines, than it is to make an a priori estimate. However, for completeness, such an a priori estimate will be made.

In order to calculate the FFT in one plane (NXN) requires

$N^2 \ln_2 N^2$ additions and,

$\frac{1}{2}N^2 \ln_2 N^2$ multiplication, or

$2N^2 \ln_2 N^2$ operations.

This must be done for N planes, requiring

$2N^3 \ln_2 N^2$ operations.

After solving the resultant set of tri-diagonal equations in transform space, the inverse must be calculated, just doubling the number of operations.

Therefore the time to take all the FFT and inverses, assuming no parallel F boxes, is

$$T_F = (4N^3 \ln_2 N^2)\ t_o \quad .$$

It appears that an F box would have an architecture similar to a V box, so, for purposes of estimate, it will be assumed that their costs are the same--i.e.,

$$C_F = C_V \quad .$$

5. The P Box

The P box solves the set of tri-diagonal equations in transform space. There are a total of N^2 sets of tri-diagonal equations to be solved. It can be shown that each set of equations requires about 13N operations and thus the complete solution requires $13N^3$ operations. The total time to do this calculation, again assuming no parallel P boxes, is

$$T_P = (13N^3)\ t_o \quad .$$

Again, it seems reasonable to assume the cost of the P box is about the same as the cost of the V box, so

$$C_P = C_V \quad .$$

6. The C Box

The C box is a control unit for the V, F, and P boxes. It would be a "hardwired" control unit which would normally sequence the boxes to execute the algorithms which calculate $\vec{V}$ and P. It would certainly be of complex design, but probably no more so than the control unit for any large general-purpose computer. Its operations would, of course, be concurrent with the other boxes and would cost nothing in processing time. Let C_c be its construction cost.

7. Memory

There are a total of N^3 mesh points. A value of $\vec{V}$ and P must be stored for each mesh point; thus at least $4N^3$ words of memory are required. If a centered-time-difference scheme is used, then $\vec{V}$ at two time steps must be stored, raising the total to $7N^3$ words of memory. It is easily seen that another $3N^3$ words is not required to store the new values of $\vec{V}$. Depending on the spatial differencing scheme, only a multiple of N^2 additional words is needed. For example, if a second-order scheme is used, only the values of $\vec{V}$ and P in the plane above and below are needed to calculate $\vec{V}$ in a particular plane. When the calculation of $\vec{V}^{n+1}$ and $\{\nabla \cdot [(\vec{V} \cdot \nabla)\vec{V}]\}^{n+1}$ for plane 2 are complete, $\vec{V}^{n-1}$ and P^n for plane 1 will never again be needed. Therefore the values of $\vec{V}^{n+1}$, for plane 1, can be stored in the memory holding $\vec{V}^{n-1}$ of plane 1 and the values of $\{\nabla \cdot [(\vec{V} \cdot \nabla)\vec{V}]\}^{n+1}$ of plane 1 can be stored in the memory holding P^n of plane 1. Thus it is necessary only to carry along two planes ($2N^2$) of $\vec{V}^{n+1}$ and $\{\nabla \cdot [(\vec{V} \cdot \nabla)\vec{V}]\}^{n+1}$ for a total of $8N^2$ additional words of memory. Because $N \gg 8$ in calculations of interest, this will be neglected in comparison to $7N^3$.

Total memory requirements are then

$$7N^3 \text{ words,}$$

and, assuming each word has 32 bits, which seems to be about the minimum acceptable, a total of

$$224N^3 \text{ bits}$$

is required.

To permit the V, F, and P boxes to operate without being held up waiting for memory fetches, it is necessary to have two words be fetched from memory in one add time. Therefore,

$$\tau_o \leq \tfrac{1}{2} t_o \quad ,$$

where τ_o is the memory cycle time.

It may not be feasible (due to cost) to provide $7N^3$ words of fast memory. If so, a buffer or cache memory of, say, 2n words with a cycle time of τ_o, and a secondary or main memory of m words with a longer cycle time of τ_1 can be used. Clearly

$$2n + m = 7N^3 \quad .$$

Pages of n words will be exchanged between the buffer and the main memories. Because $\tau_o < \tau_1$, the time required to effect the exchange will be determined by τ_1 and will be $n\tau_1$. Since the pages are being exchanged, a scheme similar to the exchange jump on the 6600 could be used to make the time $n\tau_1$ instead of $2n\tau_1$ as might be supposed.

The exchange time, $n\tau_1$, must be less then, or equal to, the time for the fastest of the boxes (V, F, or P) to process n words. The time for each of the boxes to process one mesh point (one word)

can be estimated by dividing the total process time by N^3. The minimum time per word is that of the P box and is $13t_o$. Therefore,

$$n\tau_1 \leq (13t_o)n \quad \text{or}$$

$$\tau < 13t_o \quad .$$

The cost of the memory is

$$C_M = c_o(224N^3), \quad \text{or}$$

$$= c_o(64n) + c_1(32m),$$

with

$$2n + m = 7N^3$$

c_o = cost per bit of a fast memory ($\tau_o \leq \frac{1}{2}t_o$)

c_1 = cost per bit of a slow memory ($\tau_1 \leq 13t_o$).

Note that the memory cycle times can be longer if the memory is broken up into modules. If the primary memory has four modules, then $\tau_o \leq t_o$, but, if there are eight modules, then we cannot let $\tau_o = 2t_o$ because it would still take $2t_o$ to get two words, and the processor would have to wait. However, there is a bigger advantage in using modules in the secondary memory. If there are ℓ modules in the secondary memory, then (n/ℓ) words are transmitted from each module (with cycle time τ_1) and

$$\left(\frac{n}{\ell}\right)\tau_1 \leq (13t_o)n$$

or

$$\tau_1 \leq 13\ell t_o \quad .$$

This assumes the secondary memory is a random access memory. If the secondary memory is not a random access memory but a sequential access memory (drum, disk, or tape drive), then the total transfer time is the access time, τ_a, plus the actual data transfer time and, to obviate the processor waiting,

$$\tau_a + \left(\frac{n}{\ell}\right) \tau_1 \leq (13t_o)n$$

or

$$\left(\frac{\tau_a}{n}\right) + \left(\frac{\tau_1}{\ell}\right) \leq 13t_o \quad .$$

Appendix B

A POSSIBLE APPROACH TO TURBULENCE PROBLEMS ON ILLIAC IV

As indicated in Section V, one of the main problems of doing fine-scale turbulence problems on the Illiac is that such a large number of variables is needed. There we usually considered 64-bit words, but even with 32-bit words the problem is essentially the same: Many more numbers are needed than can be held in the array memory. Accordingly, numbers have to be passed in and out of the Illiac disk several times. The problem is to put numbers to be stored after an intermediate calculation onto the disk so that those numbers which are to be used together in the next sequence of calculations are stored on the same or adjacent pages. The problem can be illustrated by considering a calculation along the lines of Method 1 of Section IV. Imagine labeling memory positions by grid position so that neighboring memory units have information from neighboring grid points. At a given time step we first compute velocity derivatives. Since these are computed using nearby values of velocities, we can compute these so that the values of these at neighboring points can be read onto the disk on the appropriate pages. At the next step we are to solve the Poisson equation for the pressure. Imagine this is a pipe problem. We first might Fourier transform in θ and z. These numbers then must go to the disk. If these numbers are denoted by $s_{ij}(r)$, what comes out at a given time are all those for a fixed r. However, for the calculation of a transform in r at the next stage, we want all $s_{ij}(r)$ for fixed (i,j) and different r all on the same page. How do we do this? Clearly this approach is not acceptable, but this may not preclude doing the calculation

by another method that does admit efficient access and egress from the disk. This may not be impossible.

To see this, we sketch a possible method of calculation which, while certainly not optimal, may perhaps work. Imagine that to solve the Poisson equation we decompose P into P_s and P_λ, where P_λ is to be a solution of the Laplace equation and P_s a particular solution of the Poisson equation corresponding to a correct condition at one part of the boundary and an assumed condition at the next layer of points within the grid. P_s is found by taking transforms in θ and z for several planes starting at a boundary surface. Then the transform $P^s_{ij}(r)$ is found essentially by solving a second order difference equation in r with prescribed initial conditions and slope. After this is found, the inverse $P^s(r)$ is found by inverting the transforms. Now the pressures at neighboring points are being produced together and can be efficiently stored in the disk. When P^s has been calculated for all r we still must satisfy the correct boundary conditions. This is done by finding a solution to the Laplace equation with boundary values which are the difference of the correct ones and those implied by our particular solution P_s.

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